MODEL ANSWER

AS-2830

B.Sc. (Hon's) (Fifth Semester) Examination, 2013

Mathematics

Numerical Analysis

Time: 3 Hrs　　　　　　　　　　　　　　　　　　Marks : 45

1. (i) $\frac{1}{2}(\Delta+\nabla) = \frac{1}{2}\left[E-1+1-E^{-1}\right]$

$$= \frac{1}{2}\left(E-E^{-1}\right)$$
$$= \mu\delta$$

(ii) Newton's forward interpolation formula

Consider a function $y=f(x)$ of an independent variable x. Let $y_0, y_1, y_2, \ldots, y_r$ be the values of y corresponding to the values $x_0, x_1, x_2, x_3, \ldots x_r$ of x respectively, then

$$y = f(x) = f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)(p-2)\cdots\boxed{p-(n-1)}}{n!}\Delta^n y_0 + \cdots$$

This formula is known as Newton's forward interpolation formula.

(iii) Divided difference table

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

(iv) Trapezoidal rule

$$y = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

Now

$$y = \int_0^4 f(x)\,dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

Here $h=1$, then

$$= \frac{1}{2}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{15}\right)\right]$$

(v) Trapezoidal rule is

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(\text{Sum of the first and last ordinates}) + 2(\text{Sum of remaining ordinates})\right]$$

(vi) The given $f(x) \equiv x^3-4x-9=0$

if $x_0 = 2 \implies f(x_0)=f(2) = 2^3-4(2)-9 = 8-8-9=-9<0$

$x_1 = 3 \implies f(x_1) = f(3) = 3^3-4(3)-9 = 27-12-9 = 27-21 = 6>0$

Therefore one of real root lies between 2 and 3

The First approximation is

$$x_2 = \frac{x_0+x_1}{2} = \frac{2+3}{2} = 2.5$$

Now $x_2 = 2.5 \implies f(x_2) = f(2.5) = (2.5)^3-4(2.5)-9$

$= -3.375 < 0$

Hence required root lies between x_2 and x_1

Now second approximation is

$$x_3 = \frac{x_1+x_2}{2} = \frac{3+2.5}{2} = 2.75$$

two roots are 2.5 and 2.75 if $x_0=2$ and $x_1=3$.

(vii) Third order Runge-Kutta method

The third order R-K method is defined

$$y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$$

where $K_1 = h f(x_0, y_0)$;

$K_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2})$

$K_3 = h f(x_0 + h, y_0 + 2K_2 - K_1)$

(viii) Eigen values are $\lambda_1 = 1$; $\lambda_2 = 2$; $\lambda_3 = 3$

(a) (i) The value of $\frac{\Delta^2}{E} x^3$ is

$$\frac{\Delta^2}{E} = \frac{(E-1)^2}{E} = \frac{E^2 - 2E + 1}{E} = E - 2 + E^{-1}$$

$$\therefore \frac{\Delta^2}{E} x^3 = (E - 2 + E^{-1}) x^3$$

$$= Ex^3 - 2x^3 + E^{-1}x^3$$

$$= (x+1)^3 - 2x^3 + (x-1)^3$$

(ii) Since $\Delta = E - 1$ then $1 + \Delta = 1 + E - 1 = E$

Therefore $(E^{\frac{1}{2}} + E^{-\frac{1}{2}})(1+\Delta)^{\frac{1}{2}} = (E^{\frac{1}{2}} + E^{-\frac{1}{2}}) E^{\frac{1}{2}}$

$= E + 1$

$= \Delta + 1 + 1$

$= 2 + \Delta$

(b) In this problem image of 0 is not given hence we may assume y_0 is x (say).

Now Difference table is

x	y_x	∇y_x	$\nabla^2 y_x$	$\nabla^3 y_x$
0	x			
		$2-x$		
1	2		$-3+x$	
		-1		$13-x$
2	1		10	
		9		
3	10			

Newton's backward difference principle is

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\cdots(p+(n-1))}{n!}\nabla^n y_n + \cdots$$

where $p = \dfrac{x-x_n}{h}$

Now, here $x=5$; $x_n = 3$; $h=1$, then

$$p = \frac{x-x_n}{h} = \frac{5-3}{1} = 2$$

$$y_n(5) = 10 + 2(9) + \frac{2(2+1)}{2!}10 + \frac{2(2+1)(2+2)}{3!}(13-x)$$

$$= 10 + 18 + 3(10) + 4(13-x)$$

$$= 110 - 4x \quad \text{for any } x \in \mathbb{R}$$

3. Let $y_0, y_1, y_2, \ldots y_n$ be the values of y corresponding to $x_0, x_1, x_2, \ldots x_n$ of x.

By definition of divided differences

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow \quad f(x) = f(x_0) + (x - x_0)\, f(x, x_0) \qquad \text{— (1)}$$

Similarly $f(x, x_0, x_1) = \dfrac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$

$$\Rightarrow \quad f(x, x_0) = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1)$$

Substituting the values of $f(x, x_1)$ in (1), we get

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x, x_0, x_1) \qquad \text{— (2)}$$

Again $f(x, x_0, x_1, x_2) = \dfrac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$

$$\Rightarrow \quad f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2)\, f(x, x_0, x_1, x_2)$$

Substituting the values of $f(x, x_0, x_1)$ in (2), we get

$$\begin{aligned} f(x) = {} & f(x_0) + (x - x_0)\, f(x_0, x_1) \\ & + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) \\ & + (x - x_0)(x - x_1)(x - x_2)\, f(x, x_0, x_1, x_2) \qquad \text{— (3)} \end{aligned}$$

Proceeding in this way, we get

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1)$$
$$+ (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$
$$+ \cdots\cdots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) f(x_0,x_1,x_2\ldots x_n)$$
$$+ (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n) f(x,x_0,x_1,x_2\ldots x_n) \quad \text{— (4)}$$

By the property of divided differences, if $f(x)$ is a polynomial of degree n, then $(n+1)$th divided difference is zero

(i.e. $f(x,x_0,x_1,\ldots x_n)=0$

Therefore equation (4) becomes

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1)$$
$$+ (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$
$$+ \cdots\cdots$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f(x_0,x_1,x_2,\ldots x_n)$$

This formula is called Newton's divided formula.

4 (a) Here we are given six values, so a polynomial of degree 5 may be fitted which ~~at~~ will have its 6^{th} difference are zero,

$$\Delta^6 f(x) = 0 \quad \forall x$$

$$\Rightarrow (E-1)^6 f(x) = 0$$

$$\Rightarrow (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0 \quad \forall x$$

$$\Rightarrow E^6 f(x) - 6E^5 f(x) + 15E^4 f(x) - 20E^3 f(x) + 15E^2 f(x) - 6E f(x) + f(x) = 0 \quad \forall x$$

$$\Rightarrow f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0 \quad \forall x \quad \text{— (1)}$$

Putting $x=1$ and 2 in (1), we get

$$f(7)-6f(6)+15f(5)-20f(4)+15f(3)-6f(2)+f(1)=0 \quad -(2)$$

and $$f(8)-6f(7)+15f(6)-20f(5)+15f(4)-6f(3)+f(2)=0 \quad -(3)$$

Substitute these values in (2) and (3), we have (after calculation,

$$15f(5)+15f(3)=2280 \Rightarrow f(5)+f(3)=152$$

$$20f(5)+6f(3)=2662 \Rightarrow 10f(5)+3f(3)=1331$$

Solve, we have $f(3)=27,\ f(5)=125.$

(b) Lagrange's interpolation

$$f(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1)$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_0-x_1)\cdots(x_0-x_{n-1})}f(x_0)$$

Now, substitute the given values, we get

$$f(x)=\frac{(x-1)(x-3)(x-5)}{(0-1)(0-3)(0-5)}f(0)+\frac{(x-0)(x-3)(x-5)}{(1-0)(1-3)(1-5)}f(1)$$

$$+\frac{(x-0)(x-1)(x-5)}{(3-0)(3-1)(3-5)}f(3)+\frac{(x-0)(x-1)(x-3)}{(5-0)(5-1)(5-3)}f(5)$$

After simplication (mathematically), we get the eq

$$f(x)=x^3-3x^2+7x-4$$

5 (a) Taking $x_0 = 24$; $x = 25$; $h = 4$, $p = \frac{x - x_0}{h} = \frac{25-24}{4} = 0.25$

Now, the central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	2854			
		308		
24	$\boxed{3162}$		$\boxed{74}$	$\boxed{-8}$
		$\boxed{382}$		
28	3544		$\boxed{66}$	
		448		
32	3992			

From the table we observe that

$$y_0 = 3162 \; ; \quad \Delta y_0 = 382$$

$$\Delta^2 y_{-1} = 74 \; ; \quad \Delta^2 y_0 = 66 \; ; \quad \Delta^3 y_{-1} = -8$$

Bessels formula is

$$y_p = \frac{y_0 + y_1}{2} + (p - \tfrac{1}{2}) \Delta y_0 + \frac{p(p-1)}{2!} \left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{(p-\frac{1}{2})\,p(p-1)}{3!} \Delta^3 y_{-1} + \cdots$$

where $p = \frac{x - x_0}{h}$

Now $$y(25) = 3162 + (0.25)(382) + \frac{(0.25)(0.25-1)}{2!}\left[\frac{74+66}{2}\right] + \frac{(0.25-0.5)(0.25)(0.25-1)}{3!}(-8)$$

$$= 3162 + 95.5 - 0.65625 - 0.0625$$

$$= 3256.7813$$

(b) This problem can solve in many ways.
One of the ways is derivatives using forward difference formula.
<u>Solution</u>:

Newton's forward difference formula is

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + (-\tfrac{1}{2})\Delta^2 y_0 + \tfrac{1}{3}\Delta^3 y_0 + (-\tfrac{1}{4})\Delta^4 y_0 + \cdots\right]$$

or

Newton's backward difference formula is

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_n + \tfrac{1}{2}\nabla^2 y_n + \tfrac{1}{3}\nabla^3 y_n + \cdots\right]$$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.105			
		0.116		
0.2	1.221		0.012	
		0.128		0.002
0.3	1.349		0.014	
		0.142		
0.4	1.491			

Here $x_n = 0.4$ and $y_n = 1.491$ and $h = 0.1$

By above formula (2), since $x = 0.4$ is at the end, we will use (2)

$$\left(\frac{dy}{dx}\right)_{x=0.4} = \frac{1}{0.1}\left[0.142 + \frac{1}{2}\,0.014 + \frac{1}{4}\,0.02\right]$$

$$= \frac{1}{0.1}\left[0.142 + 0.007 + 0.005\right] = 1.54$$

Any valid formula is also considered
or

6. General quadrature formula

Let $I = \int_a^b y\,dx$, where $y = f(x)$ takes the values $y_0, y_1, y_2, \dots y_n$ for $x_0, x_1, x_2, \dots x_n$. Let us divide the interval (a, b) into n equal parts of width h, so that

$$a = x_0,\; x_1 = x_0 + h,\; x_2 = x_0 + 2h, \dots x_n = x_0 + nh = b$$

Then $\int_{x_0}^{x_0+nh} f(x)\,dx$

Putting $x = x_0 + ph$, so that $dx = h\,dp$, in above, we get

$$I = h\int_0^n f(x_0 + ph)\,dp = h\int_0^n y_p\,dp$$

Now replacing y_p by Newton's forward interpolation formula, we get

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\right] dp$$

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Now integrating it term by term, we get after substituting the limits, as

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\Big[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\Big[\frac{n^3}{2} - \frac{n^2}{2}\Big]\Delta^2 y_0$$
$$+ \frac{1}{6}\Big[\frac{n^4}{4} - n^3 + n^2\Big]\Delta^3 y_0$$
$$+ \frac{1}{24}\Big[\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - \frac{3n^2}{1}\Big]\Delta^4 y_0$$
$$+ \cdots\cdots\Big]$$

This is called General quadrature formula

Now taking $n=3$ in above formula, we have

All differences higher than the Third will become zero and we obtain

$$\int_{x_0}^{x_3} f(x)\,dx = 3h\Big[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(6-3)}{12}\Delta^2 y_0 + \frac{3(3-2)^2}{24}\Delta^3 y_0\Big]$$

$$= 3h\Big[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\Big]$$

$$= \frac{3h}{8}\Big[y_0 + 3y_1 + 3y_2 + y_3\Big]$$

Similarly $\displaystyle\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\Big[y_3 + 3y_4 + 3y_5 + y_6\Big]$

Adding all these integrals, from x_0 to x_n, where n is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \cdots + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}\Big[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots$$
$$+ (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1})$$
$$+ 2(y_3 + y_6 + y_9 + \cdots + y_n)\Big]$$

This is called Simpson's 3/8 rule.

7 a) The derivative of y are given by

$$y' = x - y^2 ; \quad y'' = 1 - 2yy' ; \quad y''' = -2\left[(y')^2 + yy''\right]$$

$$y^{iv} = -2\left[3y'y'' + yy'''\right]$$

Here $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Now

$$y_0' = -1 , \quad y_0'' = 1 - 2(1)(-1) = 3 ; \quad y_0''' = -2\left[(-1)^2 + (1)(3)\right] = -8$$

$$y_0^{iv} = -2\left[3(-1)(3) + (1)(-8)\right] = -2\left[-9-8\right] = 34$$

By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$y_1 = y(x_1) = y(0.1) = y(0 + 0.1 \cdot 1)$$

$$x_1 = 0.1 \quad h = 0.1 \quad x_0 = 0$$

$$y_1 = y(0.1) = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2} 3 + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24} 34 + \cdots$$

$$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \cdots$$

$$= 0.91381$$

(b) Here $f(x,y) = 3x^2 + 1$, $x_0 = 1$, $y_0 = 2$

Euler's formula is

$$y_{n+1} = y_n + h f(x_n, y_n) \quad — ①$$

(i) $h = 0.5$

taking $n = 0$ in ①, we have

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 2 + 0.5 f(1, 2)$$

$$= 2 + 0.5(3 \cdot 1^2 + 1)$$

$$= 2 + 0.5(4)$$

$$= 4$$

Now $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 4 + 0.5 f(1.5, 4)$$

$$= 4 + 0.5\left[3(1.5)^2 + 1\right]$$

$$= 7.785$$

8. Let the initial approximation to the required eigen vector be $x = [1, 0, 0]'$.

Then $Ax = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$

So the first approximation to the eigenvalue is $\lambda^{(1)} = 2$

and the corresponding eigen vector $x^{(1)} = [1, -0.5, 0]'$

Hence $Ax^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)} x^{(2)}.$

Repeating the above process, we get (calculation is needed)

$$Ax^{(2)} = 2.8 \begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = 3.43 \begin{bmatrix} 0.87 \\ 1 \\ 0.54 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = 3.41 \begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = 3.41 \begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$Ax^{(6)} = 3.41 \begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)} x^{(7)}$$

clearly $\lambda^{(6)} = \lambda^{(7)}$ and $x^{(6)} = x^{(7)}$ approximately

Hence the largest eigen value is 3.41 and the corresponding eigenvector is $[0.74, -1, 0.67]^T$.

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